Write-up 11: Polar Rose Curves

David Hornbeck

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We are here going to investigate some elegant curves that result from simple functions in polar coordinates. In particular, we are going to look at *polar roses*, or curves of the following form:

$$r = a\cos(k\theta)$$

$$r = a\sin(k\theta)$$

$$r = a\cos(k\theta) + b$$

$$r = a\sin(k\theta) + b$$

$$r = \frac{c}{r = a\cos(k\theta) + b\sin(k\theta)}$$

First, let us look at varying values of a, k for $r = a\cos(k\theta)$. When $k = 1$ and we vary a, we get circles with center at $(\frac{a}{2}, 0)$ and radius $\frac{a}{2}$ as follows.

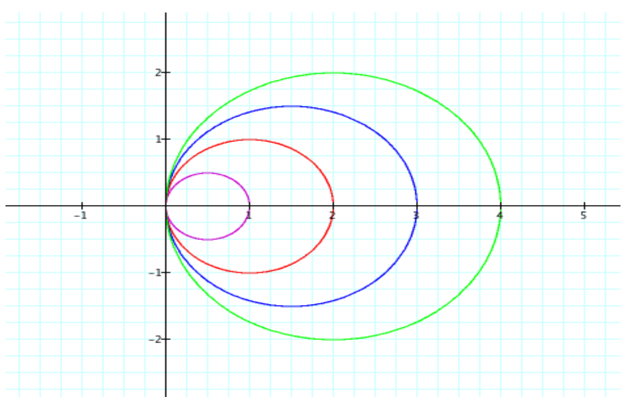


Figure 1: $r = a\cos(\theta)$ when a=1, 2, 3, 4

When a is fixed and k varies, we begin to understand the origins of the name "rose curve." Let us look at a few small integer values of k.

For $k = 2$ and $k = 4$, the graphs of $r = \cos(k\theta)$ resemble roses and have 4 and 8 pedals.

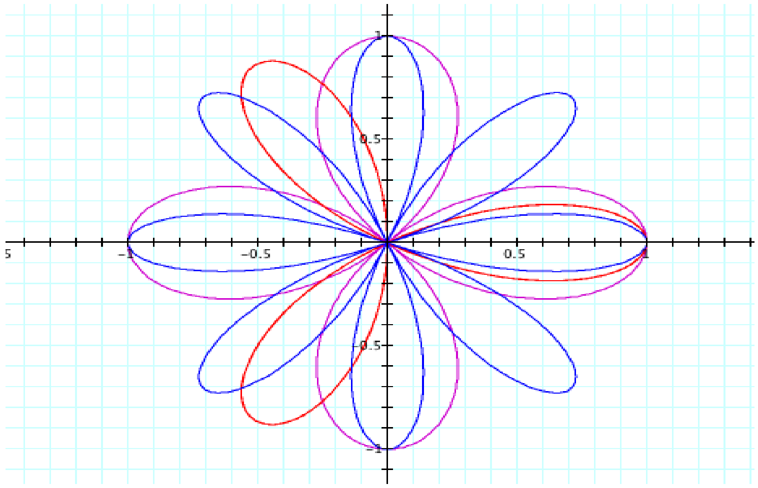


Figure 2: k = 2, 3, 4 (purple, red, blue, respectively)

This pattern continues for k even - there are $2k$ pedals on the rose. For k odd, we get a rose with k pedals.

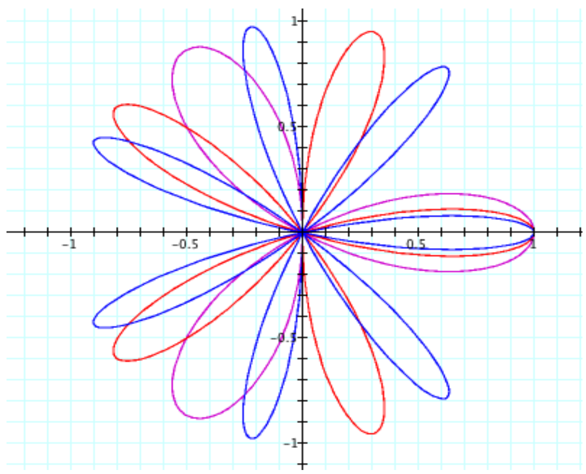
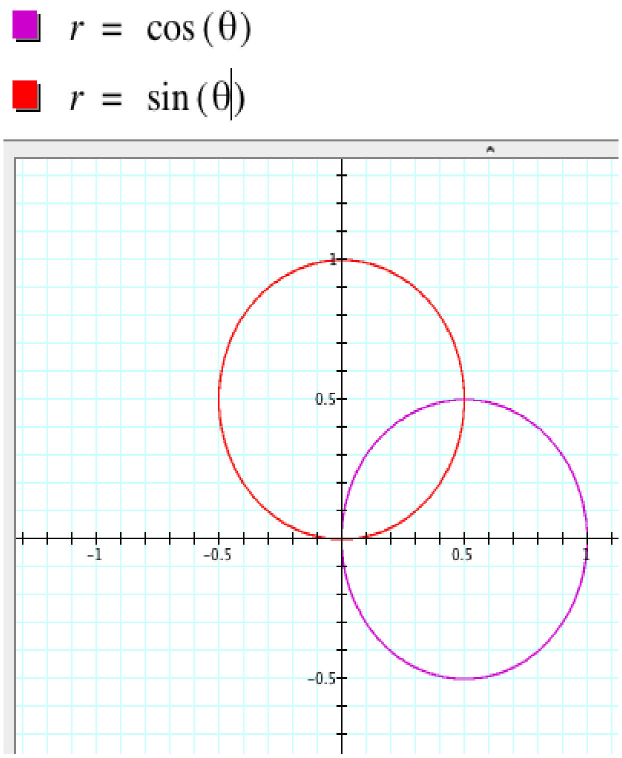
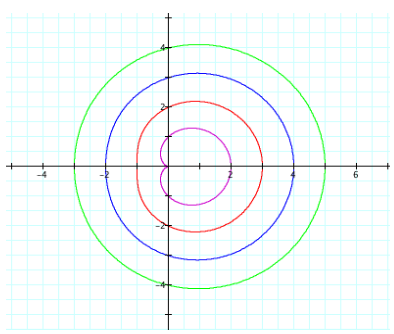


Figure 3: k= 3, 5, 7 (purple, red, blue, respectively)

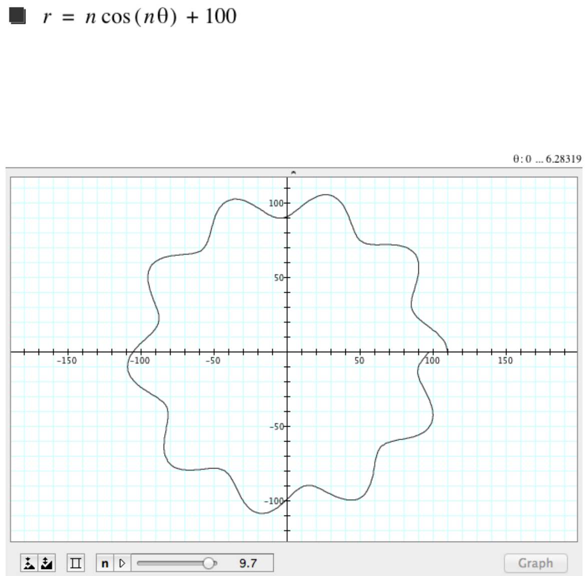
(When k is a rational fraction $\frac{m}{n}$, the rose takes an extended domain of $[0, 2n\pi]$ to develop fully. If the domain is the $[0, 2\pi]$ interval that is sufficient for $k \in \mathbb{Z}$, the graph of $a\cos(k\theta)$ will appear truncated. We will not delve into this here.) Now, rather than consider these same variables a & k for curves with equation $r = a\sin(k\theta)$, we can just observe that in polar coordinates, $\sin(\theta)$ is just a rotation of $\cos(\theta)$ by 90 degrees (as shown below for the most basic equations). Let us now examine the above equations when a



constant b is added to the trigonometric functions. We will only look at $r = a\cos(k\theta)$ for the time being, as we above observed that $\sin(\theta)$ is simply a $90°$ rotation of $\cos(\theta)$. Figure 4 shows that for $a = 1$, $k = 1$, the cosine curve approaches a circle of infinite radius as $b \to \infty$.



Similarly, when we vary b for a fixed $k \neq 1$, a larger value of b only stretches the graph of $\cos(k\theta)$ into, eventually, a theoretical circle centered at the origin with infinite radius. Observe. In particular, how $r = n\cos(n\theta) + 100$ behaves for even large n. (Note: the graph is not "complete" because $n = 9.7 \notin \mathbb{N}$.



If we fix b such that $a = b = k$, we get what is called the "*k-leaf rose*." The rose has k pedals, each with length k; the rose is symmetrical across the y-axis if k is even. Let us

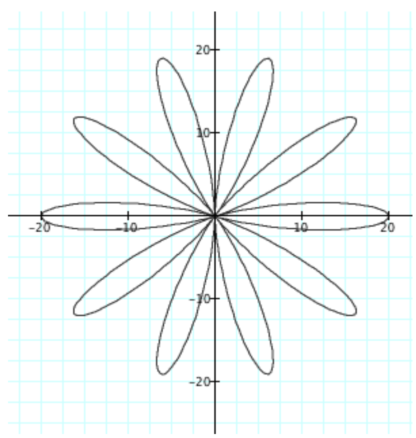


Figure 4: $r = n\cos(n\theta) + n, n = 10$

lastly look at a few examples of graphs of the form $r = \frac{c}{a\cos(k\theta)+b\sin(k\theta)}$. (Note that the presence of k ensures that the roses that would be produced by the trigonometric functions would have the same number of pedals.)

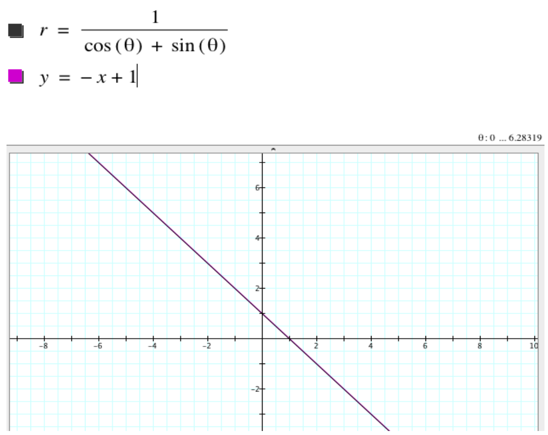


Figure 5: a=b=c=k=1

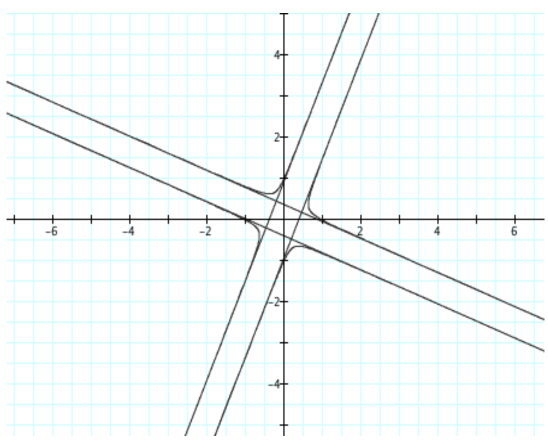


Figure 6: a=b=c=1, k=2. The graph is four lines and two hyperbolas.

It appears that whether k is even or odd again has an impact on the shape of the graph. We expect that, for $k = 3$, we will have 3 hyperbolas with a collective 3 axes of symmetry.

For variables a & b, it is intuitively clear that when $a = b$, the only impact that the size

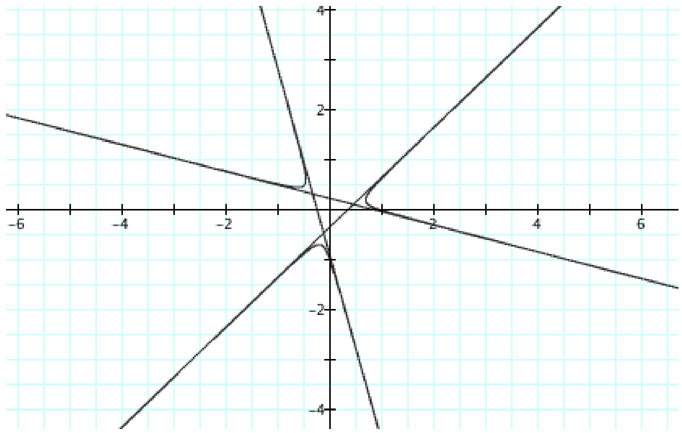


Figure 7: a=b=c=1, k=3

of a, b will have is to inversely translate the graph in a vertical direction; the larger $a(= b)$ is, r will be translated in the negative direction by $\frac{1}{n}$.

When $a \neq b$, the line $r = \frac{c}{a\cos(k\theta)+b\sin(k\theta)}$ will no longer be equivalent in polar coordinates to $y = -x + 1$, but rather $y = \frac{-a}{b}x + \frac{1}{b}(c)$. Observe an example: Lastly, let us

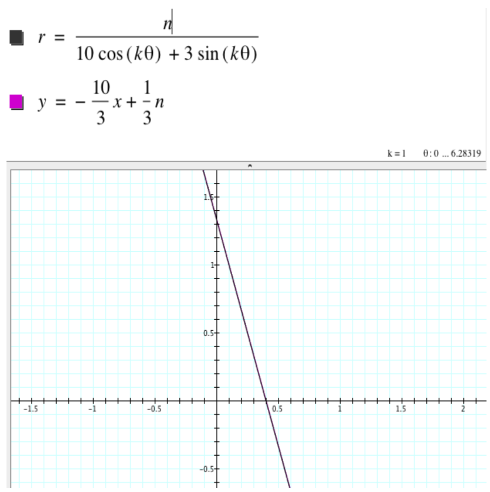


Figure 8: The two lines are perfectly coincidental.

consider all of the above considerations at once. We know that a, b determine the slope and y-intercept of the line formed by r when $k = 1$. Similarly, c determines a vertical translation of the line. When $k \neq 1$, however, we have hyperbolas that should intuitively rotate as k increases and have similar translations and stretches for varying a, b, c. Below is an example with $a = b = c = k = 7$.

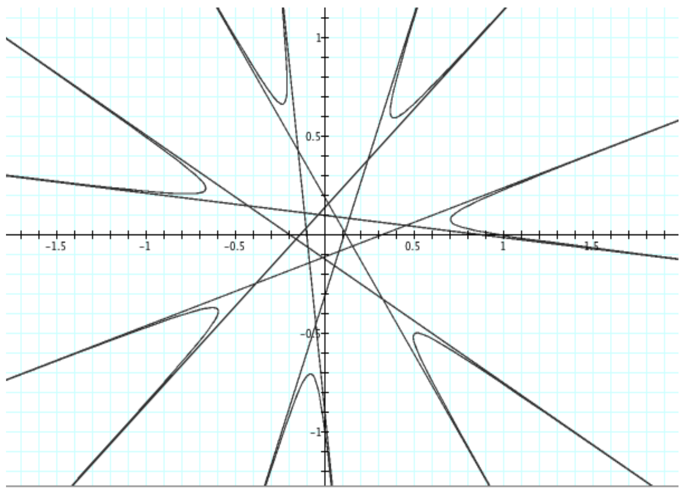


Figure 9: There are 7 axes of symmetry for 7 hybrid parabola/hyperbola curves.

Below is an animation of varying n, where $a = b = c = k = n$. The curves are indeed lovely.